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Bounds and Inequalities for General Orthogonal Polynomials on Finite Intervals

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In this paper using a new effective approach we deduce some bounds and inequalities for general orthogonal polynomials on finite intervals and give their applications to convergence of orthogonal Fourier series, Lagrange interpolation, orthogonal series with gaps, and Hermite–Fejér interpolation, as well as to the L^2 version of the principle of contamination. The main results are: we obtain far-reaching generalizations of the important results of P. Nevai on divergence of Lagrange interpolation in L^p with $p > 2$ ["Orthogonal Polynomials," Memoirs of the Amer. Math. Soc., Vol. 213, Amer. Math. Soc., Providence, RI, 1979, Corollary 10.18, p. 181; *J Approx. Theory* **43** (1985), Theorem, p. 190] and give new answers to Problems VIII and IX of P. Turán [*J. Approx. Theory* **29** (1980), pp. 32–33]; we extend Turán's Inequality [*Anal. Math.* **1** (1975), 297–311, Lemma II] to "arbitrary" measures supported in $[-1, 1]$ and solve Problem LXXI of P. Turán [*J. Approx. Theory* **29** (1980), p. 71].

1. INTRODUCTION

Let $\alpha(x)$ be a nondecreasing function on $[-1, 1]$ with infinitely many points of increase such that all moments of $d\alpha(x)$ are finite and $\{P_n(x)\}$,

$$P_n(x) := P_n(d\alpha, x) = \gamma_n x^n + \dots \tag{1}$$

($\gamma_n := \gamma_n(d\alpha) > 0$), the orthonormal polynomials with respect to $d\alpha$. The support of $d\alpha$ is the set of points of increase of $\alpha(x)$ and is denoted by $\text{supp}(d\alpha)$. We call $d\alpha$ a measure.

Denote a triangular matrix of nodes by

$$X: (x_0 \equiv x_{0,n} \equiv) 1 \geqslant x_{1n} > x_{2n} > \cdots > x_{nn} \geqslant -1 (\equiv x_{n+1,n} \equiv x_{n+1}), \qquad n = 1, 2, \dots, \tag{2}$$

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and the Lagrange interpolating polynomial of $f \in C[-1, 1]$ by

$$L_n(f) := L_n(X, f) := L_n(X, f, x) := \sum_{k=1}^{n} f(x_{kn})\, l_{kn}(x), \qquad n = 1, 2, ...,$$

where the fundamental polynomials

$$l_{kn}(x) = \frac{\omega_n(x)}{(x - x_{kn})\, \omega_n'(x_{kn})}, \qquad k = 1, 2, ..., n, \quad n = 1, 2, ...,$$

with $\omega_n(x) = (x - x_{1n})(x - x_{2n}) \cdots (x - x_{nn})$, $n = 1, 2, ...$ If X consists of the zeros of $P_n(x)$ then we write $L_n(d\alpha, f)$ instead of $L_n(X, f)$. For simplicity sometimes we also write x_k instead of x_{kn}, etc.

As we know,

$$\lambda_n(x) := \lambda_n(d\alpha, x) = \sum_{k=0}^{n-1} P_k^2(x) = \sum_{k=1}^{n} \frac{l_{kn}^2(x)}{\lambda_{kn}}, \qquad n = 1, 2, ... \tag{3}$$

are called the Christoffel functions, where

$$\lambda_{kn} = \lambda_n(x_{kn}), \qquad k = 1, 2, ..., n, \quad n = 1, 2, ...$$

In attempting to study convergence of orthogonal Fourier series or convergence of Lagrange interpolation at zeros of orthogonal polynomials, one invariably encounters the need for bounds and inequalities on the orthogonal polynomials on the interval of orthogonality. Historically, the problem of finding bounds and inequalities has lived under the shadow of the deeper asymptotics on the segment, for the latter are often the only way of obtaining the former. Of course, this way usually gives asymptotic estimates for certain "nice" measures only. In this paper we develop an effective approach to find bounds and inequalities of many important quantities in orthogonal polynomials for general measures on finite intervals. Thus this makes it possible to extend many important results previously obtained. In particular, we obtain far-reaching generalizations of the important results of P. Nevai on divergence of Lagrange interpolation in L^p with $p > 2$ [11, Corollary 10.18, p. 181; 12, Theorem, p. 190] and give new answers to Problems VIII and IX of P. Turán [18, pp. 32–33]; we extend Turán's Inequality [17, Lemma II] to "arbitrary" measures supported in $[-1, 1]$ and solve Problem LXXI of P. Turán [18, p. 71].

In the next section we state a basic theorem on which our approach is based. Then in Section 3 we give bounds and inequalities for γ_{n-1}/γ_n, $\sum \lambda_{kn}^q\, |P_{n-1}(x_{kn})|^p$, and $P_n(x)$, respectively. Finally, in Section 4, we discuss some applications of these results to convergence of orthogonal Fourier series, convergence of Lagrange interpolation, orthogonal series with gaps and Hermite–Fejér interpolation, as well as to the L^2 version of the principle of contamination.

2. The Basic Theorem

To state our result we need to introduce some notations. For each n, $n = 1, 2, \ldots$, define $1 \leqslant i_n \leqslant j_n \leqslant n$, $m_n = j_n - i_n + 1$, and $\Delta_n := [\xi_{j_n}, \xi_{i_n}]$ where

$$\xi_{i_n} = \begin{cases} x_{i_n} & \text{for} \quad i_n > 1 \\ 1 & \text{for} \quad i_n = 1, \end{cases}$$

$$\xi_{j_n} = \begin{cases} x_{j_n} & \text{for} \quad j_n < n \\ -1 & \text{for} \quad j_n = n. \end{cases}$$

Then we have

Theorem 1. *Let s be a fixed positive integer, and let $1 \leqslant i_n \leqslant j_n \leqslant n$ for $n = 1, 2, \ldots$. Then for any sequence of positive numbers $\mathscr{E} = \{\varepsilon_n\}$ and for any matrix X there exist sets*

$$I_n := I_n(\mathscr{E}, s, X, \Delta_n) = \bigcup_{k = i_n}^{j_n} (x_{kn} - h_{kn}, x_{kn} + h'_{kn}) \cap \Delta_n$$

with $h_{kn}, h'_{kn} > 0$ such that $|I_n| \leqslant \varepsilon_n$ and

$$\Lambda_n(x) := \sum_{k = i_n}^{j_n} |(x - x_{kn})\, l_{kn}(x)|^s \geqslant \frac{\varepsilon_n^s m_n^{1-s}}{(24)^s} \tag{4}$$

holds for all $x \in \Delta_n \backslash I_n$ and $n = 1, 2, \ldots$. Moreover, the order $\varepsilon_n^s m_n^{1-s}$ is the best possible and is attained by the Chebyshev nodes.[1]

Proof. Main ideas of the proof, including the origin of the crucial lemma below, can be found in the important paper of P. Vértesi [19]. The proof is analogous to that of Theorem 1 in [15, p. 763] and corrects some technical constants to fill a minor gap there.

If $|\Delta_n| = 0$ or $\varepsilon_n \geqslant |\Delta_n|$ then there is nothing to prove, so assume that $|\Delta_n| > 0$ and $\varepsilon_n < |\Delta_n|$. Let n be fixed. We introduce the following notations, where $d, d_k \geqslant 0$, and adopt the convention that $[a, b] := \phi$ if $a > b$:

[1] The last conclusion and its proof are due to one of the referees. I proved that the order m_n^{1-s} is the best possible and is attained by the Chebyshev nodes.

$$J_k = [x_{k+1}, x_k], \qquad k = 0, 1, ..., n;$$

$$J_k(d) = \begin{cases} [x_{k+1} + d, x_k - d], & k = 1, 2, ..., n-1 \\ [x_1 + d, x_0], & k = 0 \\ [x_{n+1}, x_n - d], & k = n; \end{cases}$$

$$\bar{J}_k := \overline{J_k(d_k)} := J_k \backslash J_k(d_k), \qquad k = 0, 1, ..., n;$$

$$M_n := \{k : J_k \subset \Delta_n\};$$

$$N_n := M_n \backslash \{0, n\}.$$

Our proof is based on a result of P. Erdös and P. Turán [4, Lemma IV]:

$$l_k(x) + l_{k+1}(x) \geqslant 1, \qquad x \in J_k, \quad k = 1, 2, ..., n-1$$

$$l_1(x) \geqslant 1, \qquad x \in J_0$$

$$l_n(x) \geqslant 1, \qquad x \in J_n.$$

A brief outline of the proof is: For a given ε_n we will give I_n in the form

$$I_n = \bigcup_{k \in M_n} \bar{J}_k = \bigcup_{k \in M_n} J_k \backslash J_k(d_k),$$

and then determine these d_k's and estimate the measures of these $\bar{J}_k$'s.

To this end we distinguish two cases according to $k \in M_n \cap \{0, n\}$ or $k \in N_n$.

Case 1. $k \in M_n \cap \{0, n\}$.

Put $d_0 = d_n = 2^{-3} \varepsilon_n m_n^{(1-s)/s}$. If, e.g., $|J_0| > 0$ and $x \in J_0(d_0)$ we use $l_1(x) > 1$ to deduce

$$\Lambda_n(x) \geqslant |x - x_1|^s \, |l_1(x)|^s \geqslant d_0^s = 8^{-s} \varepsilon_n^s m_n^{1-s} \geqslant c \varepsilon_n^s m_n^{1-s}$$

and

$$|\bar{J}_0 \cup \bar{J}_n| \leqslant d_0 + d_n \leqslant \frac{\varepsilon_n}{4}.$$

Case 2. $k \in N_n$.

Obviously, if $N_n \neq \phi$ then $i_n < j_n$, since $i_n = j_n$ implies $|\Delta_n| = 0$ or $i_n = j_n = 1$ or $i_n = j_n = n$, each of which means $N_n = \phi$. Define $z_k = z_k(d_k)$ by

$$|\omega_n(z_k)| = \min_{x \in J_k(d_k)} |\omega_n(x)|. \tag{5}$$

Here we refer to a result and its proof given by the author in [15, Lemma, p. 764] which provides a lower bound for two arbitrary successive terms in the sum $\Lambda_n(x)$.

LEMMA. *Let $s \geqslant 1$, $n \geqslant 2$, and $1 \leqslant k, r \leqslant n-1$. Then*

$$L_k(x) + L_{k+1}(x) \geqslant 2^{1-s} d_k^s \left| \frac{\omega_n(z_r)}{\omega_n(z_k)} \right|^s, \qquad x \in J_r(d_r),$$

where

$$L_k(x) = |(x - x_k)\, l_k(x)|^s, \qquad k = 1, 2, ..., n.$$

Proof of the lemma. Since

$$L_i(x) = \left| \frac{\omega_n(x)}{\omega_n'(x_i)} \right|^s = \left| \frac{\omega_n(x)}{\omega_n(z_r)} \right|^s L_i(z_r),$$

by (5) we have

$$L_i(x) \geqslant L_i(z_r), \qquad x \in J_r(d_r), \quad i = k, k+1.$$

So for $x \in J_r(d_r)$

$$\begin{aligned} L_k(x) + L_{k+1}(x) &\geqslant L_k(z_r) + L_{k+1}(z_r) \\ &= \left| \frac{\omega_n(z_r)}{\omega_n(z_k)} \right|^s (|z_k - x_k|^s\, |l_k^s(z_k)| + |z_k - x_{k+1}|^s\, |l_{k+1}^s(z_k)|) \\ &\geqslant d_k^s \left| \frac{\omega_n(z_r)}{\omega_n(z_k)} \right|^s (|l_k^s(z_k)| + |l_{k+1}^s(z_k)|) \\ &\geqslant 2^{1-s} d_k^s \left| \frac{\omega_n(z_r)}{\omega_n(z_k)} \right|^s \end{aligned}$$

using $l_k(x) + l_{k+1}(x) \geqslant 1$ for $x \in J_k$ to deduce that $l_k^s(x) + l_{k+1}^s(x) \geqslant 2^{1-s}$. This completes the proof of the lemma. ∎

Now we continue the proof of Theorem 1. Let d_k be defined by

$$d_k = \sup\{d\text{: there exists a point } x \in J_k(d) \text{ such that (4) does not hold}\}, \qquad k \in N_n.$$

Obviously $0 < d_k \leqslant \frac{1}{2}|J_k|$ and (4) holds for all points, except at most m_n isolated points, in the set $\bigcup_{k \in N_n} J_k(d_k)$. Thus it suffices to estimate the measure $v_n := |\bigcup_{k \in N_n} \bar{J}_k| = |\bigcup_{k \in N_n} J_k \backslash J_k(d_k)|$. Omitting those intervals for which $d_k \leqslant q_n := v_n/6m_n$ we shall estimate the measure $\mu_n := |\bigcup_{k \in K_n} \bar{J}_k|$, where $K_n := \{k \in N_n : d_k > q_n\}$. Since by definition

$$v_n = 2 \sum_{k \in N_n} d_k, \qquad \mu_n = 2 \sum_{k \in K_n} d_k \tag{6}$$

and

$$\nu_n - \mu_n = \left| \bigcup_{k \in N_n \setminus K_n} \bar{J}_k \right| \leqslant 2m_n q_n \leqslant \frac{\nu_n}{3},$$

we have $2\nu_n/3 \leqslant \mu_n$. Thus, if we can show

$$\mu_n \leqslant \tfrac{1}{2}\varepsilon_n, \tag{7}$$

then

$$\nu_n \leqslant \tfrac{3}{4}\varepsilon_n \tag{8}$$

and $|I_n| \leqslant |\bar{J}_0 \cup \bar{J}_n| + \nu_n \leqslant \varepsilon_n$, where

$$I_n = \bigcup_{k \in M_n} \bar{J}_k = \bigcup_{k = i_n}^{j_n} (x_k - d_k, x_k + d_{k-1}) \cap \Delta_n,$$

which will complete the proof.

Now we will prove (7). By the definition of the d_r's for each $r \in N_n$ we can choose a point $u_r \in J_r((\frac{2}{3})^{1/s} d_r)$ (since $(\frac{2}{3})^{1/s} < 1$) so that (4) does not hold, i.e.,

$$A_n(u_r) < c\varepsilon_n^s m_n^{1-s}, \qquad \forall r \in N_n, \tag{9}$$

where $c = (24)^{-s}$. Thus by (6) we have

$$\sum_{r \in K_n} d_r A_n(u_r) < 2^{-1} c\mu_n \varepsilon_n^s m_n^{1-s}. \tag{10}$$

On the other hand, let $z_k = z_k((\frac{2}{3})^{1/s} d_k)$ be defined in (5), i.e.,

$$|\omega_n(z_k)| = \min_{x \in J_k((2/3)^{1/s} d_k)} |\omega_n(x)|.$$

Noting that $j_n \notin N_n$ and hence $j_n \notin K_n$, by the lemma we obtain

$$\begin{aligned} d_r A_n(u_r) &= d_r \sum_{k = i_n}^{j_n} L_k(u_r) \\ &\geqslant \frac{1}{2} d_r \sum_{k = i_n}^{j_n - 1} [L_k(u_r) + L_{k+1}(u_r)] \\ &\geqslant \frac{1}{2} d_r \sum_{k \in K_n} [L_k(u_r) + L_{k+1}(u_r)] \\ &\geqslant \frac{1}{2} d_r \sum_{k \in K_n} 2^{1-s} \left[\left(\frac{2}{3}\right)^{1/s} d_k\right]^s \left|\frac{\omega_n(z_r)}{\omega_n(z_k)}\right|^s \\ &> 3^{-1} 2^{1-s} q_n^{s-1} \sum_{k \in K_n} d_r d_k \left|\frac{\omega_n(z_r)}{\omega_n(z_k)}\right|^s, \end{aligned}$$

where the last inequality follows from the fact that $d_k > q_n$ for $k \in K_n$. Hence

$$\sum_{r\in K_n} d_r \Lambda_n(u_r) > 3^{-1}2^{1-s}q_n^{s-1} \sum_{r\in K_n}\sum_{k\in K_n} d_r d_k \left|\frac{\omega_n(z_r)}{\omega_n(z_k)}\right|^s. \tag{11}$$

Using the inequality $t+1/t \geqslant 2$ $(t>0)$ and (6) we obtain

$$\begin{aligned}\sum_{r\in K_n}\sum_{k\in K_n} d_r d_k \left|\frac{\omega_n(z_r)}{\omega_n(z_k)}\right|^s &\geqslant 2^{-1}\sum_{r\in K_n}\sum_{\substack{k\geqslant r\\ k\in K_n}} d_r d_k \left(\left|\frac{\omega_n(z_r)}{\omega_n(z_k)}\right|^s + \left|\frac{\omega_n(z_k)}{\omega_n(z_r)}\right|^s\right)\\ &\geqslant \sum_{r\in K_n}\sum_{\substack{k\geqslant r\\ k\in K_n}} d_r d_k\\ &\geqslant 2^{-1}\sum_{r\in K_n}\sum_{k\in K_n} d_r d_k = 2^{-3}\mu_n^2.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{r\in K_n} d_r \Lambda_n(u_r) &> 3^{-1}2^{-s-2}q_n^{s-1}\mu_n^2\\ &\geqslant 3^{-1}2^{-s-2}6^{1-s}\mu_n^{s+1}m_n^{1-s} = 2^{s-1}c\mu_n^{s+1}m_n^{1-s}.\end{aligned}$$

In comparison with (10) one obtains (7). Since c does not depend on n , the proof of (4) is complete.

For the Chebyshev nodes it is well known that

$$L_k(x) = \{n^{-1}(1-x_k^2)^{1/2}\,|T_n(x)|\}^s, \qquad k=1,2,...,n,$$

where $T_n(x)$ denotes the Chebyshev polynomial of first kind. If we assume, say, that $\frac{1}{4}n \leqslant k \leqslant \frac{3}{4}n$ and $d_k = \frac{1}{4}\,|J_k|$ then

$$L_k(x) \sim n^{-s}, \qquad x \in \bigcup_{r=0}^{n} J_r(d_r).$$

Meanwhile, for i_n and j_n with $\frac{1}{4}n \leqslant i_n < j_n \leqslant \frac{3}{4}n$ we have $|\Delta_n| \sim \varepsilon_n \sim m_n/n$. Thus for $x \in \bigcup_{r=0}^n J_r(d_r) \cap \Delta_n$ one obtains

$$\Lambda_n(x) = \sum_{k=i_n}^{j_n} L_k(x) \sim \frac{m_n}{n^s} \sim \varepsilon_n^s m_n^{1-s}. \quad \blacksquare$$

Remark. Theorem 1 in [15, p. 762] mentioned above is a special case when $\Delta_n \equiv [-1, 1]$. In this paper we usually use the conclusion for $s=1$ and in this case (4) becomes

$$\Lambda_n(x) = \sum_{k=i_n}^{j_n} |(x-x_{kn})\,l_{kn}(x)| \geqslant \frac{\varepsilon_n}{24}. \tag{12}$$

Theorem 1 provides a method to estimate lower bounds of many important quantities in orthogonal polynomials and approximation theory. This method of estimates is suitable for every matrix of zeros (1), for every measure $d\alpha$ defined on $[-1, 1]$, and usually for every $n \in \mathbb{N}$. But other approaches for estimates previously used usually give asymptotic estimation for certain "good" measures only.

3. Bounds and Inequalities

3.1. Write

$$\begin{aligned}
Z(\alpha') &:= \{x \in [-1, 1]: \alpha'(x) = 0\};\\
\mathcal{M} &:= \text{the collection of all Lebesgue measurable sets in } [-1, 1];\\
|\Omega| &:= \text{the measure of } \Omega, \qquad \Omega \in \mathcal{M};\\
\sigma(\Delta;\delta) &:= \sigma(d\alpha;\Delta;\delta) := \frac{\inf_{\Omega \in \mathcal{M},\, \Omega \subset \Delta,\, |\Omega| = \delta} \int_\Omega d\alpha(x)}{\int_\Delta d\alpha(x)},\\
&\qquad \Delta \in \mathcal{M},\ 0 < \delta \leqslant |\Delta|;\\
\sigma(\delta) &:= \sigma([-1, 1]; \delta).
\end{aligned} \tag{13}$$

Definition. We write $\alpha \in \Sigma$ if there exists a $\delta < 2$ such that $\sigma(d\alpha; [-1, 1]; \delta) > 0$.

Now we have

Lemma 1. *If* $|\Delta| \geqslant \delta > |\Delta \cap Z(\alpha')|$ *then* $\sigma(\Delta;\delta) > 0$. *In particular, if* $|Z(\alpha')| < 2$, *then* $\alpha \in \Sigma$.

Proof. Write

$$E_n = \left\{x \in \Delta : \alpha'(x) \leqslant \frac{1}{n}\right\}.$$

Then we claim that for every $\varepsilon > |\Delta \cap Z(\alpha')|$ there exists N such that $|E_N| < \varepsilon$. In fact, suppose to the contrary that there would be $\varepsilon_0 > |\Delta \cap Z(\alpha')|$ satisfying $|E_n| \geqslant \varepsilon_0$ for all n. It is clear that $E_{n+1} \subset E_n$, $n = 1, 2, \ldots$ and hence for $E = \bigcap_{n=1}^{\infty} E_n$ we have

$$|E| = \lim_{n \to \infty} |E_n| \geqslant \varepsilon_0.$$

On the other hand, if $x \in E$ then $x \in \Delta \cap Z(\alpha')$, which is a contradiction.

Now choose ε so that $\delta > \varepsilon > |\Delta \cap Z(\alpha')|$. As proved before, there exists an $N = N(\varepsilon)$ such that $|E_N| < \varepsilon$. Then for any Ω with $\Omega \in \mathcal{M}$, $\Omega \subset \Delta$, and $|\Omega| = \delta$ we have

$$\int_\Omega d\alpha(x) \geqslant \int_\Omega \alpha'(x)\, dx \geqslant \int_{\Omega \setminus E_N} \alpha'(x)\, dx \geqslant \frac{\delta - \varepsilon}{N}.$$

Whence $\sigma(\Delta;\delta) \geqslant ((\delta - \varepsilon)/N)\{\int_\Delta d\alpha(x)\}^{-1} > 0$. ∎

The main result in this subsection is

THEOREM 2. *Let $d\alpha$ be an arbitrary measure supported in $[-1, 1]$. Then for every $0 < \delta < 2$,*

$$\frac{\gamma_{n-1}}{\gamma_n} \sum_{k=1}^{n} \lambda_{kn} |P_{n-1}(x_{kn})| \geqslant \frac{(2-\delta)\,\sigma(\delta)\{\int_{-1}^{1} d\alpha(x)\}^{1/2}}{24}, \quad n = 1, 2, \ldots. \tag{14}$$

Proof. Our starting point is the expression [11, p. 6]

$$P_n(x) \frac{\gamma_{n-1}}{\gamma_n} \lambda_{kn} P_{n-1}(x_{kn}) = (x - x_{kn})\, l_{kn}(x),$$
$$k = 1, 2, \ldots, n, \quad n = 1, 2, \ldots, \tag{15}$$

from which it follows that for each $n = 1, 2, \ldots,$

$$\begin{aligned}\int_{-1}^{1} &\sum_{k=1}^{n} |(x - x_k)\, l_k(x)|\, d\alpha(x) \\ &= \int_{-1}^{1} |P_n(x)|\, d\alpha(x) \frac{\gamma_{n-1}}{\gamma_n} \sum_{k=1}^{n} \lambda_k |P_{n-1}(x_k)| \\ &\leqslant \left\{\int_{-1}^{1} d\alpha(x)\right\}^{1/2} \frac{\gamma_{n-1}}{\gamma_n} \sum_{k=1}^{n} \lambda_k |P_{n-1}(x_k)|.\end{aligned}$$

Thus,

$$\frac{\gamma_{n-1}}{\gamma_n} \sum_{k=1}^{n} \lambda_k |P_{n-1}(x_k)| \geqslant \frac{\int_{-1}^{1} \sum_{k=1}^{n} |(x - x_k)\, l_k(x)|\, d\alpha(x)}{\{\int_{-1}^{1} d\alpha(x)\}^{1/2}}.$$

Applying Theorem 1 with $s = 1$, $\Delta_n \equiv [-1, 1]$, and $\varepsilon_n \equiv 2 - \delta$, we get I_n such that $|I_n| \leqslant 2 - \delta$ and

$$\begin{aligned}\frac{\gamma_{n-1}}{\gamma_n} \sum_{k=1}^{n} \lambda_k |P_{n-1}(x_k)| &\geqslant \frac{((2-\delta)/24) \int_{[-1,1] \setminus I_n} d\alpha(x)}{\{\int_{-1}^{1} d\alpha(x)\}^{1/2}} \\ &\geqslant \frac{(2-\delta)\,\sigma(\delta)\{\int_{-1}^{1} d\alpha(x)\}^{1/2}}{24}. \quad ∎\end{aligned}$$

Combining Theorem 2 and Lemma 1 we easily get

COROLLARY 1. *For every measure $d\alpha$ supported in $[-1, 1]$*

$$\frac{\gamma_{n-1}}{\gamma_n} \geq \frac{(2-\delta)\,\sigma(\delta)}{24}, \qquad n = 1, 2, ...,$$

and for $\alpha \in \Sigma$

$$\liminf_{n \to \infty} \frac{\gamma_{n-1}}{\gamma_n} > 0. \tag{16}$$

COROLLARY 2. *Let $\alpha \in \Sigma$. Then*

$$\lim_{n \to \infty} \left\{ \sum_{k=1}^{n} \frac{1}{|P_n'(x_{kn})|} \right\}^{1/n} = 1. \tag{17}$$

Proof. It follows from Theorem 2 and Lemma 1 that

$$\liminf_{n \to \infty} \frac{\gamma_{n-1}}{\gamma_n} \sum_{k=1}^{n} \lambda_k \, |P_{n-1}(x_k)| > 0. \tag{18}$$

Obviously

$$\frac{\gamma_{n-1}}{\gamma_n} \sum_{k=1}^{n} \lambda_k \, |P_{n-1}(x_k)| \leq \left\{ \sum_{k=1}^{n} \lambda_k \right\}^{1/2} = \left\{ \int_{-1}^{1} d\alpha(x) \right\}^{1/2}. \tag{19}$$

On the other hand, (15) implies

$$\frac{\gamma_{n-1}}{\gamma_n} \lambda_k \, |P_{n-1}(x_k)| = \frac{1}{|P_n'(x_k)|}, \qquad k = 1, 2, ..., n, \quad n = 1, 2, ... \tag{20}$$

Thus our conclusion can be deduced from (18–20). ∎

3.2. According to an inequality given by G. Freud in [6, formula (24)]

$$\sum_{k=1}^{n} \frac{\lambda_{kn} P_{n-1}^2(x_{kn})}{1 - x_{kn}^2} \leq 2 \left(\frac{\gamma_n}{\gamma_{n-1}} \right)^2 \tag{21}$$

and using (16) one obtains an estimate as follows.

In what follows, the sign "O" depends only on α.

THEOREM 3. *Let $\alpha \in \Sigma$. Then*

$$\sum_{k=1}^{n} \frac{\lambda_{kn} P_{n-1}^2(x_{kn})}{1 - x_{kn}^2} = O(1). \tag{22}$$

As immediate consequences of Theorem 3 as well as (15) we state the following corollaries.

COROLLARY 3. *Let $\alpha \in \Sigma$. Then for every number $0 < \varepsilon_n \leqslant 1$,*

$$\sum_{1 - x_{kn}^2 \leqslant \varepsilon_n} \lambda_{kn} P_{n-1}^2(x_{kn}) = O(\varepsilon_n). \tag{23}$$

COROLLARY 4. *Let $\alpha \in \Sigma$. Then*

$$\sum_{k=1}^{n} \frac{\lambda_{kn} \, |P_{n-1}(x_{kn})|}{(1 - x_{kn}^2)^{1/2}} = O(1), \tag{24}$$

$$\sum_{k=1}^{n} \frac{\int_{-1}^{1} (x - x_{kn})^2 \, l_{kn}^2(x) \, d\alpha(x)}{\lambda_{kn}(1 - x_{kn}^2)} = O(1), \tag{25}$$

$$\sum_{k=1}^{n} \frac{\int_{-1}^{1} |(x - x_{kn}) \, l_{kn}(x)| \, d\alpha(x)}{(1 - x_{kn}^2)^{1/2}} = O(1), \tag{26}$$

and

$$\sum_{k=1}^{n} \frac{\int_{-1}^{1} |(x - x_{kn}) \, l_{kn}^2(x)| \, d\alpha(x)}{\lambda_{kn}^{1/2}(1 - x_{kn}^2)^{1/2}} = O(1). \tag{27}$$

Combining (20) and (22) gives

COROLLARY 5. *Let $\alpha \in \Sigma$. Then*

$$\sum_{k=1}^{n} \frac{1}{\lambda_{kn} P_n'(x_{kn})^2 \, (1 - x_{kn}^2)} = O(1). \tag{28}$$

As to lower bounds we have

THEOREM 4. *Let $\Delta \subset [-1, 1]$ be a union of finitely many disjoint intervals. If $p \geqslant 1$ and*

$$\int_{\Delta} \alpha'(x) \, dx > 0, \tag{29}$$

then

$$\liminf_{n \to \infty} \sum_{x_{kn} \in \Delta} \lambda_{kn} \, |P_{n-1}(x_{kn})|^p > 0. \tag{30}$$

Moreover, (29) is necessary for validity of (30) provided that $p < 2$ and α is absolutely continuous.

Proof. Since Δ is a union of finitely many disjoint intervals, (29) implies that there exists an interval $\Omega \subset \Delta$ such that $\int_\Omega \alpha'(x)\,dx > 0$. So we may suppose without loss of generality that $\Delta = [a, b]$ and that if $\int_c^d \alpha'(x)\,dx = \int_a^b \alpha'(x)\,dx$ for $a \leqslant c \leqslant d \leqslant b$ then $c = a$ and $d = b$.

Now let δ satisfy $0 < \delta \leqslant \frac{1}{4} |\Delta \backslash Z(\alpha')|$ and

$$\int_{a+\delta}^{b-\delta} \alpha'(x)\,dx \geqslant \tfrac{1}{4} \int_\Delta \alpha'(x)\,dx.$$

Hence by Theorem 6.1.1 in [16, p. 107] there exists a number N such that for every $n \geqslant N$, $P_n(x)$ has at least one zero in both $[a, a+\delta]$ and $[b-\delta, b]$, say, x_{j_n} and x_{i_n}, respectively. Then $a \leqslant x_{j_n} \leqslant a+\delta$, $b-\delta \leqslant x_{i_n} \leqslant b$, and

$$\int_{x_{j_n}}^{x_{i_n}} \alpha'(x)\,dx \geqslant \tfrac{1}{4} \int_\Delta \alpha'(x)\,dx$$

whenever $n \geqslant N$. Now we need the following lemma the proof of which is similar to that of Theorem 2 and, therefore, it is omitted.

LEMMA 2. *For every $1 \leqslant i_n \leqslant j_n \leqslant n$ and for any $0 < \delta_n < |\Delta_n|$, where Δ_n is defined in Theorem* 1, *we have*

$$\sum_{k=i_n}^{j_n} \lambda_{kn} |P_{n-1}(x_{kn})| \geqslant \frac{\delta_n \sigma(\Delta_n; |\Delta_n| - \delta_n)\{\int_{\Delta_n} d\alpha(x)\}^{1/2}}{24}, \quad n = 1, 2, ..., \tag{31}$$

and for every $1 \leqslant i_n \leqslant j_n \leqslant n$ and for any $0 < \delta_n < |\Omega_n|$, where $\Omega_n := [x_{j_n}, x_{i_n}]$, we have

$$\sum_{k=i_n}^{j_n} \lambda_{kn} |P_{n-1}(x_{kn})| \geqslant \frac{\delta_n \sigma(\Omega_n; |\Omega_n| - \delta_n)\{\int_{\Omega_n} d\alpha(x)\}^{1/2}}{24}, \quad n = 1, 2, ... \tag{32}$$

Now we continue the proof of Theorem 4. Applying Lemma 2 we have

$$\begin{aligned}\sum_{x_k \in \Delta} \lambda_k |P_{n-1}(x_k)| &\geqslant \sum_{k=i_n}^{j_n} \lambda_k |P_{n-1}(x_k)| \\ &\geqslant \frac{\delta\sigma(\Omega_n; |\Omega_n| - \delta)\{\int_{\Omega_n} d\alpha(x)\}^{1/2}}{24} \\ &\geqslant \frac{\delta\sigma(\Delta; |\Delta| - 3\delta)\{\int_\Delta \alpha'(x)\,dx\}^{1/2}}{48}\end{aligned}$$

for $n \geqslant N$. Since

$$|\Delta| - 3\delta \geqslant |\Delta| - \tfrac{3}{4} |\Delta \backslash Z(\alpha')| > |\Delta| - |\Delta \backslash Z(\alpha')| = |\Delta \cap Z(\alpha')|,$$

by Lemma 1 we have that $\sigma(\Delta; |\Delta| - 3\delta) > 0$ and

$$\liminf_{n\to\infty} \sum_{x_k \in \Delta} \lambda_k |P_{n-1}(x_k)| > 0. \tag{33}$$

Using Hölder's Inequality, (30) follows from (33).

To prove the second part of Theorem 4, suppose to the contrary that $\int_\Delta \alpha'(x)\,dx = 0$ where Δ is a nonempty interval. By absolute continuity $\int_\Delta d\alpha(x) = 0$. Applying Theorem 2.4 in [5, p. 18] we see that for every n, Δ contains no more than one zero of $P_n(x)$. Therefore for $p < 2$

$$\lambda_k |P_{n-1}(x_k)|^p = \lambda_k^{1-p/2} [\lambda_k P_{n-1}^2(x_k)]^{p/2} \leqslant \lambda_k^{1-p/2} \to 0$$

as $n \to \infty$. Here we apply an identity $\sum_{k=1}^n \lambda_k P_{n-1}^2(x_k) = 1$ to deduce $\lambda_k P_{n-1}^2(x_k) \leqslant 1$ for every k and n, and use the result that for absolutely continuous measures [14, p. 46]

$$\lim_{n\to\infty} \max_{1\leqslant k\leqslant n} \lambda_{kn} = 0. \quad \blacksquare$$

Theorem 4, together with Lemma 1, gives

COROLLARY 6. *Let $\alpha \in \Sigma$ and let Δ be defined as in Theorem 4. If $p \geqslant 1$ then* (30) *holds for any Δ satisfying*

$$|\Delta| > |Z(\alpha')|. \tag{34}$$

We can also obtain the following estimate of lower bounds for $\sum_{k=1}^n \lambda_k^2 P_{n-1}^2(x_k)$, which may easily be deduced from Theorem 1 with $s = 2$. We omit the proof.

THEOREM 5. *Let $d\alpha$ be an arbitrary measure. Then for every $0 < \delta < 2$*

$$\sum_{k=1}^n \lambda_{kn}^2 P_{n-1}^2(x_{kn}) \geqslant \frac{(2-\delta)^2 \sigma(\delta) \int_{-1}^1 d\alpha(x)}{576n}, \qquad n = 1, 2, \ldots. \tag{35}$$

3.3. As an immediate consequence of (18) and (35) we have

THEOREM 6. *Let $\alpha \in \Sigma$. Then*

$$|P_n(x)| = O(1) \sum_{k=1}^n |(x - x_{kn})\, l_{kn}(x)| \tag{36}$$

and

$$P_n^2(x) = O(n) \sum_{k=1}^n (x - x_{kn})^2\, l_{kn}^2(x). \tag{37}$$

COROLLARY 7. *Let $\alpha \in \Sigma$. Then*

(a) *The statement*

$$|P_n(x)| = O(1) \tag{38}$$

is equivalent to

$$\sum_{k=1}^{n} |(x - x_{kn})\, l_{kn}(x)| = O(1); \tag{39}$$

(b)
$$|P_n(x)| = O(1) \sum_{k=1}^{n} |l_{kn}(x)|; \tag{40}$$

(c)
$$\lambda_n(x)\, P_n^2(x) = O(1); \tag{41}$$

(d)
$$\frac{\lambda_n(x)}{\lambda_{n+1}(x)} = O(1). \tag{42}$$

Proof. (a) Equation (36) gives the "if" portion and (15) yields the "only if" portion.

(b) Equation (40) follows from (36).

(c) By (40) we have

$$P_n^2(x) = O(1) \left\{ \sum_{k=1}^{n} |l_k(x)| \right\}^2 = O(1) \left\{ \sum_{k=1}^{n} \lambda_k \right\} \left\{ \sum_{k=1}^{n} \frac{l_k^2(x)}{\lambda_k} \right\} = O(1)\, \lambda_n^{-1}(x)$$

and (41) follows.

(d) Use (41) and the identity

$$\frac{\lambda_n(x)}{\lambda_{n+1}(x)} = 1 + \lambda_n(x)\, P_n^2(x). \quad \blacksquare$$

Now we turn to discuss lower bounds for $P_n(x)$. The following is a Turán type inequality with an effective expression and generalizes all results on finite intervals previously known, including those of Turán [17, p. 307] and Máté *et al.*[10, p. 279].

THEOREM 7. *Let $d\alpha$ and $d\beta$ be arbitrary measures supported in $[-1, 1]$ and $0 < p < \infty$. Then for every $\Delta \in \mathcal{M}$ the inequalities*

$$\int_\Delta |P_n(d\alpha, x)|^p \, d\beta(x)$$
$$\geqslant \frac{(|\Delta| - \delta)^p \, \sigma(d\beta; \Delta; \delta) \int_\Delta d\beta(x)}{(24)^p \left(\int_{-1}^{1} d\alpha(x)\right)^{p/2}}, \quad n = 0, 1, \ldots \tag{43}$$

and

$$\int_{\Delta} |P_n(d\alpha, x)|^p \, d\beta(x) \geqslant \frac{(|\Delta| - \delta)^p \, \sigma(d\beta; [-1, 1]; \delta) \int_{-1}^{1} d\beta(x)}{(24)^p \left(\int_{-1}^{1} d\alpha(x)\right)^{p/2}}, \quad n = 0, 1, \ldots \tag{44}$$

hold, whenever $\delta < |\Delta|$. *In particular, if* $|Z(\alpha')| < 2$, *then for every* $2 \geqslant \delta > |Z(\alpha')|$,

$$\int_{\Delta} P_n^2(d\alpha, x) \, d\alpha(x) \geqslant \frac{(\delta - |Z(\alpha')|)^2 \, \sigma((\delta + |Z(\alpha')|)/2)}{2304} > 0, \quad n = 0, 1, \ldots \tag{45}$$

holds, whenever $|\Delta| > \delta$.

Proof. Applying Theorem 1 with $s = 1$ and $\varepsilon_n \equiv |\Delta| - \delta$ we can choose I_n so that $|I_n| \leqslant \varepsilon_n = |\Delta| - \delta$ and

$$\sum_{k=1}^{n} |(x - x_k) \, l_k(x)| \geqslant \frac{|\Delta| - \delta}{24}, \qquad x \in [-1, 1] \backslash I_n, \quad n = 1, 2, \ldots.$$

On the other hand, by (15) we get

$$\int_{\Delta \backslash I_n} |P_n(d\alpha, x)|^p \, d\beta(x) \left\{ \frac{\gamma_{n-1}}{\gamma_n} \sum_{k=1}^{n} \lambda_k \, |P_{n-1}(d\alpha, x_k)| \right\}^p = \int_{\Delta \backslash I_n} \left\{ \sum_{k=1}^{n} |(x - x_k) \, l_k(x)| \right\}^p d\beta(x).$$

Obviously,

$$\left\{ \frac{\gamma_{n-1}}{\gamma_n} \sum_{k=1}^{n} \lambda_k \, |P_{n-1}(d\alpha, x_k)| \right\}^p \leqslant \left\{ \int_{-1}^{1} d\alpha(x) \right\}^{p/2}.$$

It follows from (13) that $\int_{\Delta \backslash I_n} d\beta \geqslant \sigma(d\beta; \Delta; \delta) \int_{\Delta} d\beta$. Thus

$$\begin{aligned} \int_{\Delta} |P_n(d\alpha, x)|^p \, d\beta(x) &\geqslant \int_{\Delta \backslash I_n} |P_n(d\alpha, x)|^p \, d\beta(x) \\ &\geqslant \frac{(|\Delta| - \delta)^p \int_{\Delta \backslash I_n} d\beta(x)}{(24)^p \left(\int_{-1}^{1} d\alpha(x)\right)^{p/2}} \\ &\geqslant \frac{(|\Delta| - \delta)^p \, \sigma(d\beta; \Delta; \delta) \int_{\Delta} d\beta(x)}{(24)^p \left(\int_{-1}^{1} d\alpha(x)\right)^{p/2}}. \end{aligned}$$

Similarly, using $\int_{\Delta \setminus I_n} d\beta \geqslant \sigma(d\beta; [-1, 1]; \delta) \int_{-1}^{1} d\beta$ deduced from (13), (44) follows. Inequality (45) follows from (44) if we put $\beta = \alpha$ and $p = 2$, and replace δ by $(\delta + |Z(\alpha')|)/2$. ∎

This important theorem has a number of immediate consequences.

COROLLARY 8. *If* $|\Delta \setminus Z(\beta')| > 0$, *i.e.*,

$$\int_{\Delta} \beta'(x)\, dx > 0,$$

then

$$\liminf_{n \to \infty} \int_{\Delta} |P_n(d\alpha, x)|^p\, d\beta(x) > 0. \tag{46}$$

Moreover, if β *is absolutely continuous on* Δ *then the converse is true. In particular,*

$$\liminf_{n \to \infty} \int_{\Delta} |P_n(d\alpha, x)|^p\, dx > 0 \tag{47}$$

if and only if $|\Delta| > 0$.

COROLLARY 9. *If* $\int_{-1}^{1} d\alpha(x) = 1$ *and* $0 < p < \infty$, *then for every* $\Delta \in \mathcal{M}$

$$\int_{\Delta} |P_n(d\alpha, x)|^p\, dx \geqslant \frac{|\Delta|^{p+1}}{2(48)^p}, \qquad n = 0, 1, ... \tag{48}$$

Proof. Take $\delta = \frac{1}{2}|\Delta|$ and $\beta \equiv 1$ in (43). ∎

Using Theorem 1 and (15) we can also prove

THEOREM 8. *For every measure* $d\alpha$ *and for any sequence of positive numbers* $\mathscr{E} = \{\varepsilon_n\}$ *there are sets*

$$I_n := I_n(\mathscr{E}, d\alpha) = \bigcup_{k=1}^{n} (x_{kn} - h_{kn}, x_{kn} + h'_{kn}) \cap [-1, 1]$$

with $h_{kn}, h'_{kn} > 0$ *such that* $|I_n| \leqslant \varepsilon_n$ *and*

$$|P_n(d\alpha, x)| \geqslant \frac{\varepsilon_n}{24(\int_{-1}^{1} d\alpha(x))^{1/2}}, \tag{49}$$

holds for all $x \in [-1, 1] \setminus I_n$ *and* $n = 1, 2, ...$

We conclude this section by formulating the following

THEOREM 9. *Let $\alpha \in \Sigma$. If α is continuous at $x \in [-1, 1]$, then*

$$\lim_{n \to \infty} \lambda_n(x) P_n^2(x) = 0 \tag{50}$$

if and only if

$$\lim_{n \to \infty} \lambda_n(x) \left\{ \sum_{k=1}^{n} |l_{kn}(x)| \right\}^2 = 0. \tag{51}$$

Proof. If (51) holds, then (50) follows according to (40). Conversely, apply Lemmas 9.1 and 10.1 in [11, pp. 156, 175] and use continuity of α at x. ∎

4. APPLICATIONS

The above bounds and inequalities, especially, (16), (30), and (46), make it possible to extend many important results previously obtained on convergence of orthogonal Fourier series and on mean convergence of Lagrange interpolation. The reason is that the inequalities of the forms (16), (30), and (46), and so forth, are just the crucial points to prove these results and these inequalities were previously obtained for certain "nice" measures only, since only for "nice" measures can one use asymptotics to get them. Sometimes we will not give detailed proofs and will give the inequalities needed and the references quoted only, since only small modifications are necessary.

4.1. *Absolute Convergence of Orthogonal Fourier Series*

THEOREM 10. *Let $d\alpha$ be an arbitrary measure. Then the series*

$$\sum_{n=0}^{\infty} |c_n P_n(d\alpha, x)| \tag{52}$$

either diverges almost everywhere in $[-1, 1]$ or converges almost everywhere in the set $\{x \in [-1, 1]: \alpha'(x) > 0\}$, and the latter case is equivalent to

$$\sum_{n=0}^{\infty} |c_n| < \infty. \tag{53}$$

Proof. By the same arguments as in Theorem 3 in [9, p. 321] the convergence of (52) on a set $E \subset [-1, 1]$, $|E| > 0$, implies (53), since that proof only applies the inequality

$$\liminf_{n \to \infty} \int_{\Delta} |P_n(d\alpha, x)| \, dx > 0$$

for every set Δ with $|\Delta| > 0$. But in the present case by (47) it is true for every measure $d\alpha$. On the other hand, by Lebesgue's *Monotone Convergence Theorem* it follows from (53) that

$$\begin{aligned}\int_{-1}^{1} \sum_{n=0}^{\infty} |c_n P_n(x)| \, \alpha'(x) \, dx &\leqslant \int_{-1}^{1} \sum_{n=0}^{\infty} |c_n P_n(x)| \, d\alpha(x) \\ &\leqslant \sum_{n=0}^{\infty} |c_n| \int_{-1}^{1} |P_n(x)| \, d\alpha(x) \\ &\leqslant \sum_{n=0}^{\infty} |c_n| \left\{ \int_{-1}^{1} d\alpha(x) \right\}^{1/2} < \infty,\end{aligned}$$

which means $\sum_{n=0}^{\infty} |c_n P_n(x)| \, \alpha'(x) < \infty$ holds almost everywhere in $[-1, 1]$, and hence $\sum_{n=0}^{\infty} |c_n P_n(x)| < \infty$ holds almost everywhere in the set $\{x \in [-1, 1] : \alpha'(x) > 0\}$. ∎

Remark. This theorem is a far-reaching extension of Theorem 3 in [9, p. 321], which is proved for measures $d\alpha$ with $\alpha' > 0$, a.e.

4.2. *Mean Convergence of Orthogonal Fourier Series*

For $f \in L^1_{d\alpha}$ the nth partial sum of its orthogonal Fourier series in orthogonal polynomials $P_k(d\alpha)$ is defined by

$$S_n(d\alpha, f, x) = \sum_{k=0}^{n-1} P_k(x) \int_{-1}^{1} f(t) \, P_k(t) \, d\alpha(t). \tag{54}$$

Now we formulate the following important result in which we use the notation

$$\|g\|_{v\,d\alpha,\,p} = \left\{ \int_{-1}^{1} |g(t)|^p \, v(x) \, d\alpha(x) \right\}^{1/p}.$$

THEOREM 11. *Let* $0 < p < \infty$, $1 < q < \infty$, $q' = q/(q-1)$, *and* $u^{1/(1-q)}, u, w \in L^1_{d\alpha}$. *Let* $\int_{-1}^{1} w(x) \, \alpha'(x) \, dx > 0$ *and* $\int_{-1}^{1} u(x) \, \alpha'(x) \, dx > 0$. *Suppose*

$$\|S_n(d\alpha, f)\|_{w\,d\alpha,\,p} \leqslant C \, \|f\|_{u\,d\alpha,\,q}, \qquad n = 1, 2, \ldots, \quad \forall f \in L^q_{u\,d\alpha}, \tag{55}$$

with C independent of n and f. Then

$$\sup_{n \geqslant 1} \|P_n(d\alpha)\|_{w\,d\alpha,\,p} < \infty \tag{56}$$

and

$$\sup_{n \geqslant 1} \|P_n(d\alpha)\, u^{-1}\|_{u\,d\alpha,\,q'} < \infty. \tag{57}$$

Moreover, if $p \geqslant 2$ *then*

$$\int_{-1}^{1} [\alpha'(x)(1-x^2)^{1/2}]^{-p/2}\, w(x)\, \alpha'(x)\, dx < \infty \tag{58}$$

and if $1 < q \leqslant 2$ *then*

$$\int_{-1}^{1} [\alpha'(x)(1-x^2)^{1/2}]^{q/2(1-q)}\, u^{1/(1-q)}(x)\, \alpha'(x)\, dx < \infty. \tag{59}$$

Proof. First we point out that using Hölder's Inequality, $f \in L^q_{u\,d\alpha}$, together with $u^{1/(1-q)} \in L^1_{d\alpha}$, implies $f \in L^1_{d\alpha}$. Now (55) gives

$$\|S_{n+1}(d\alpha, f) - S_n(d\alpha, f)\|_{w\,d\alpha,\,p} \leqslant C\, \|f\|_{u\,d\alpha,\,q},$$

i.e.,

$$\|P_n\|_{w\,d\alpha,\,p} \left| \int_{-1}^{1} f P_n\, d\alpha(x) \right| \leqslant C\, \|f\|_{u\,d\alpha,\,q}. \tag{60}$$

If we choose $f(x) = (|P_n(x)|\, u^{-1}(x))^{1/(q-1)}$ sign $P_n(x)$ then $f \in L^q_{u\,d\alpha}$ so that by (60) we have $\|P_n\|_{w\,d\alpha,\,p}\, \|P_n u^{-1}\|_{u\,d\alpha,\,q'} \leqslant C$. Applying Corollary 8 we conclude $\inf_{n \geqslant 1} \|P_n\|_{w\,d\alpha,\,p} > 0$ and $\inf_{n \geqslant 1} \|P_n u^{-1}\|_{u\,d\alpha,\,q'} > 0$. Hence (56) and (57) follow. Now (58) and (59) follow from Theorem 7.31 in [11, p. 138]. ▮

COROLLARY 10. *Let* $d\alpha$ *be an arbitrary measure supported in* $[-1, 1]$. *If*

$$\int_{-1}^{1} [\alpha'(x)^{\varepsilon}\, (1-x^2)^{1/2}]^{-1}\, dx = \infty$$

for every $\varepsilon > 0$, *then the sequence of Fourier operators* $\{S_n(d\alpha)\}$ *is not uniformly bounded in* $L^p_{d\alpha}$ *provided* $1 \leqslant p \leqslant \infty$ *and* $p \neq 2$.

Proof. For $p = 1$ or $p = \infty$ the corollary is true (see [11, p. 169]).

Now assume $1 < p < \infty$ and $p \neq 2$. In Theorem 11 let $q = p \neq 2$, $w = u = 1$. Then it follows from (58) and (59) that

$$\int_{-1}^{1} \alpha'(x)^{1-p/2} (1-x^2)^{-p/4}\, dx < \infty, \qquad \text{if} \quad p > 2 \tag{61}$$

and

$$\int_{-1}^{1} \alpha'(x)^{1-p/2(p-1)} (1-x^2)^{-p/4(p-1)}\, dx < \infty, \qquad \text{if} \quad p < 2. \tag{62}$$

On the other hand, by the assumptions, neither (61) nor (62) could be true. ∎

Remark. The special examples applying Corollary 10 can be found in [11, Corollary 8.14, p. 155; Theorems 9.26 and 9.27, pp. 168–169].

4.3. *Mean Convergence of Lagrange Interpolation and Problems* VIII *and* IX *of P. Turán*

A well-known result proved by Erdös and Turán in [3] is

THEOREM A. *For every function* $f \in C[-1, 1]$

$$\lim_{n \to \infty} \int_{-1}^{1} |f(x) - L_n(d\alpha, f, x)|^2\, d\alpha(x) = 0.$$

It is natural to ask whether one can obtain conditions guaranteeing

$$\lim_{n \to \infty} \int_{-1}^{1} |f(x) - L_n(d\alpha, f, x)|^p\, w(x)\, dx = 0.$$

for all $f \in C[-1, 1]$, where $0 < p < \infty$ and $w\ (\geqslant 0) \in L^1[-1, 1]$. Freud lists this problem as unsolved problem No. 1 in his book [5, p. 273]. Meanwhile, one of Turán's favorite and frequently repeated problems was the following [18, p. 32]

Problem VIII. Does there exist an absolutely continuous measure $d\alpha$ with support in $[-1, 1]$ such that, for some $f \in C[-1, 1]$, we have

$$\limsup_{n \to \infty} \int_{-1}^{1} |f(x) - L_n(d\alpha, f, x)|^p\, d\alpha(x) = \infty \tag{63}$$

for every $p > 2$?

It is remarkable that both Freud and Turán agreed that the resolution of this problem is of primary significance. In connection with Problem VIII

Turán also proposed its somewhat weaker form and another problem [18, pp. 33–34], which are restated as follows.

Problem IX. Does there exist an absolutely continuous measure $d\alpha$ with support in $[-1, 1]$ such that, for every given $p > 2$, there is an $f \in C[-1, 1]$, such that (63) holds?

Problem XI. Given $p > 1$, what is a necessary and sufficient condition that

$$\lim_{n \to \infty} \int_{-1}^{1} |f(x) - L_n(X, f, x)|^p \, dx = 0 \tag{64}$$

for every $f \in C[-1, 1]$?

Askey [1, p. 77] conjectured that the Pollaczek weight [11, p. 80] solves Turán's Problems VIII and IX, and Nevai proved it in [11, Corollary 10.18, p. 181; 12, Theorem, p. 190].

In what follows, S is the Szegö class, i.e., $\alpha \in S$ means $\operatorname{supp}(d\alpha) = [-1, 1]$ and

$$\frac{\log \alpha'(x)}{\sqrt{1 - x^2}} \in L^1[-1, 1],$$

and JS (just Szegö) denotes the set of α satisfying $\alpha \in S$ and

$$\frac{[\alpha'(x)]^{-\varepsilon}}{\sqrt{1 - x^2}} \notin L^1[-1, 1]$$

holds for every $\varepsilon > 0$.

Nevai's results are the following.

THEOREM B. *Let either $\alpha \in JS$ or let α be a Pollaczek weight or let α be defined by*

$$\alpha'(x) = \varphi(x) \exp\{-(1 - x^2)^{-1/2}\},$$

where φ $(> 0) \in \operatorname{Lip} 1$. Then for every $p > 2$ there exists a function $f \in C[-1, 1]$ such that (63) *holds.*

THEOREM C. *Let $\alpha \in S$, $1 \leqslant p_0 < \infty$, and w $(\geqslant 0) \in L^1[-1, 1]$. Suppose that*

$$\int_{-1}^{1} [\alpha'(x)(1 - x^2)^{1/2}]^{-p/2} \, w(x) \, dx = \infty \tag{65}$$

holds for every $p > p_0$. *Then there exists a function* $f \in C[-1, 1]$ *such that*

$$\limsup_{n \to \infty} \int_{-1}^{1} |L_n(d\alpha, f, x)|^p \, w(x) \, dx = \infty \tag{66}$$

for every $p > p_0$.

Hence, Theorem C solves Problem VIII if we set $w = \alpha'$. The crucial points to prove Theorems B and C are to show (16) and to show that there exists a number $\delta > 0$ such that (33) holds for any Δ with $|\Delta| > \delta$, where $\Delta \subset [-1, 1]$ is a union of finitely many disjoint intervals. The first problem can be solved easily for the measures in Theorems B and C, since by Rahmanov's Theorem [13, Theorem 4.5.7, p. 20] we have

$$\lim_{n \to \infty} \frac{\gamma_{n-1}}{\gamma_n} = \frac{1}{2}.$$

As to the second one, Nevai points out in [13, p. 29] that it is even more difficult and he can only prove the following results [11, Corollary 9.13, p. 161 and Theorem 9.10, p. 160].

THEOREM D. *Let* $\alpha'(x)$ $(\in L^1[-1, 1]) > 0$ *for almost every* $x \in (-1, 1)$ *and* $\Delta \subset [-1, 1]$. *Let the sequence* $\{|P_n(x)|\}$ *be uniformly bounded for* $x \in \Delta$. *Then* (33) *is valid.*

THEOREM E. *Let* $\alpha \in S$. *Then there exists a number* $\delta = \delta(d\alpha) > 0$ *such that if* $\Delta \subset [-1, 1]$ *is a union of finitely many disjoint intervals with* $|\Delta| \geqslant 2 - \delta$ *then* (33) *is valid.*

Now Corollary 1 and Theorem 4 make it possible to extend Theorems B and C in case $2 < p_0 < \infty$ to a measure $\alpha \in \Sigma$ (see the definition of Σ before Lemma 1).

THEOREM 12. *Let* $\alpha \in \Sigma$. *If* w $(\geqslant 0) \in L^1$ *and* $0 < p \leqslant \infty$, *then*

$$\|P_n(d\alpha)\|_{w,p} \leqslant C \, \|L_n(d\alpha)\|_{L^\infty \to L^p_w}, \tag{67}$$

where $C = C(d\alpha)$ *and*

$$\|L_n(d\alpha)\|_{L^\infty \to L^p_w} = \sup_{\|f\|_\infty = 1} \|L_n(d\alpha, f)\|_{w,p}.$$

Proof. The conclusion in case $p = \infty$ is given by (40). Now assume $0 < p < \infty$. Let $\delta = 1 - \frac{1}{2}|Z(\alpha')|$ and let an interval $\Omega \subset [-1, 1]$ satisfy $|\Omega| = \delta$. Then we can choose two intervals τ_1 and τ_2 and define $\Delta = \tau_1 \cup \tau_2$

so that $\tau_1 \cap \tau_2 = \phi$, $\operatorname{dist}(\Omega, \Delta) > 0$, and $|\Delta| > |Z(\alpha')|$. Then we obtain exactly in the same way as in Theorem 10.15 in [11, p. 180] that

$$\|I_\Omega P_n\|_{w,p} \frac{\gamma_{n-1}}{\gamma_n} \sum_{x_k \in \Delta} \lambda_k |P_{n-1}(x_k)| \leqslant 2 \|L_n(d\alpha)\|_{L^\infty \to L^p_w},$$

where I_Ω is the characteristic function of Ω. By (16) and (30) there exists a number $d(\Delta, d\alpha) > 0$ and $N(\Delta)$ such that

$$\frac{\gamma_{n-1}}{\gamma_n} \sum_{x_k \in \Delta} \lambda_k |P_{n-1}(x_k)| \geqslant d^{-1}(\Delta, d\alpha)$$

whenever $n \geqslant N(\Delta)$. Thus for $n \geqslant N(\Delta)$ one has

$$\|I_\Omega P_n\|_{w,p} \leqslant 2d(\Delta, d\alpha) \|L_n(d\alpha)\|_{L^\infty \to L^p_w}. \tag{68}$$

Of course, we can choose a number $d_1(\Delta, d\alpha)$ instead of $2d(\Delta, d\alpha)$ so that (68) holds for all $n \geqslant 0$. Since this inequality holds for any $\Omega \subset [-1, 1]$ with $|\Omega| = \delta > 0$, (67) must hold as well. ∎

COROLLARY 11. *Let $\alpha \in \Sigma$ and w $(\geqslant 0) \in L^1$. If $0 < p < \infty$ and*

$$\limsup_{n \to \infty} \|L_n(d\alpha)\|_{L^\infty \to L^p_w} < \infty \tag{69}$$

then

$$\limsup_{n \to \infty} \|P_n(d\alpha)\|_{w,p} < \infty. \tag{70}$$

Moreover, if $p \geqslant 2$ and (69) holds then

$$\int_{-1}^{1} [\alpha'(x)(1-x^2)^{1/2}]^{-p/2} w(x)\, dx < \infty. \tag{71}$$

Proof. Applying Theorem 12 and Theorem 7.31 in [11, p. 138], (71) follows from (70). ∎

This corollary extends Theorem 10.16 in [11, p. 181] and the conclusion of Theorem 10.15 for $p \geqslant 2$ in [11, p. 180]. Letting $w = \alpha'$ and $w = 1$, respectively, from Corollary 11 and the uniform boundedness principle we immediately obtain the following two important results.

COROLLARY 12. *Let α be absolutely continuous and let $2 < p < \infty$. If α satisfies*

$$\int_{-1}^{1} [\alpha'(x)(1-x^2)^{1/2}]^{-p/2} \alpha'(x)\, dx = \infty, \tag{72}$$

then there exists a function $f \in C[-1, 1]$ *such that*

$$\limsup_{n \to \infty} \int_{-1}^{1} |f(x) - L_n(d\alpha, f, x)|^p \, d\alpha(x) = \infty. \tag{73}$$

COROLLARY 13. *Let* $\alpha \in \Sigma$ *and let* $p > 2$. *If*

$$\lim_{n \to \infty} \int_{-1}^{1} |f - L_n(d\alpha, f)|^p \, dx = 0 \tag{74}$$

for all $f \in C[-1, 1]$ *then*

$$\int_{-1}^{1} [\alpha'(x)(1 - x^2)^{1/2}]^{-p/2} \, dx < \infty. \tag{75}$$

Obviously, Corollary 12 extends Theorem B, since all the weights in Theorem B satisfy (72) for each $p > 2$. Meanwhile, Corollary 13 gives a partial answer to Problem XI, more exactly, a necessary condition such that (64) holds for all $f \in C[-1, 1]$ in case $L_n(X, f) = L_n(d\alpha, f)$. In addition, using Corollary 11, we can also prove

COROLLARY 14. *Let* $\alpha \in \Sigma$. *Let* $w\ (\geqslant 0) \in L^1[-1, 1]$ *and* $2 \leqslant p_0 < \infty$. *If*

$$\int_{-1}^{1} [\alpha'(x)(1 - x^2)^{1/2}]^{-p/2} \, w(x) \, dx = \infty$$

holds for every $p > p_0$, *then there exists a function* $f \in C[-1, 1]$ *such that*

$$\limsup_{n \to \infty} \int_{-1}^{1} |L_n(d\alpha, f, x)|^p \, w(x) \, dx = \infty \tag{76}$$

holds for every $p > p_0$.

Proof. Use Corollary 11 and the technical proposition given by Nevai in [13, Theorem 4.8.3, p. 45]. ▌

This corollary gives a generalization of Theorem 4.8.2 in [13, p. 44] and new negative answers to Problems VIII and IX of Turán in [18, pp. 32–33].

4.4. *Orthogonal Series with Gaps and Problem* LXXI *of P. Turán*

P. Turán in [17, Lemma II] proved the following important inequality for a measure $\alpha \in S$ (A. Máté *et al.* in [10, Theorem 13.3] extended it to the measures $d\alpha$ satisfying $\alpha'(x) > 0$ a.e. in $[-1, 1]$).

LEMMA A. *Let $\alpha \in S$, and let*

$$-1 \leqslant b-\delta < b+\delta \leqslant 1. \tag{77}$$

Then the inequality

$$\int_{b-\delta}^{b+\delta} P_n^2(x)\, d\alpha(x) \geqslant C_1(d\alpha, \delta) \tag{78}$$

holds uniformly for $n = 1, 2, \ldots,$ and b satisfying(77).

He also pointed out that $C_1(d\alpha, \delta)$ here is an ineffective expression depending only upon δ and $d\alpha$, that is, it cannot be calculated explicitly, and it would be of special relevance to replace it by an effective one. The background of this requirement was explained in [17; 18, Sect. 59]. In a few words, inequality (78) plays a crucial role in developing some results of P. Turán and N. Wiener, and to give the quantities in these results explicitly we need the explicit expression for $C_1(d\alpha, \delta)$.

As a method to solve this problem, later P. Turán in [18, p. 71] further proposed the following

Problem LXXI. Give an explicit estimate for $n_0(d\alpha, \delta)$ such that, if $\alpha \in S$, then

$$\int_{b-\delta}^{b+\delta} P_n^2(x)\, d\alpha(x) > \frac{1}{2\pi} \int_{b-\delta}^{b+\delta} \frac{dx}{\sqrt{1-x^2}} > \frac{\delta}{\pi}, \tag{79}$$

holds for $n > n_0(d\alpha, \delta)$.

Since for $n \leqslant n_0(d\alpha, \delta)$ we can determine a quantity $C_0(d\alpha, \delta)$ such that for all permitted b's

$$\int_{b-\delta}^{b+\delta} P_n^2(x)\, d\alpha(x) \geqslant C_0(d\alpha, \delta) > 0,$$

the quantity $C_1(d\alpha, \delta)$ occurring in (78) can be chosen so that

$$C_1(d\alpha, \delta) = \min\left\{C_0(d\alpha, \delta), \frac{\delta}{\pi}\right\},$$

which is an effective expression, although it cannot be calculated easily.

Now if we put $\beta = \alpha$, $p = 2$, and $\delta = \frac{1}{2}|\Delta| > |Z(\alpha')|$, then (44) gives the following result which extends (78) to general measures and provides a satisfactory solution to Problem LXXI of Turán, i.e., an explicit expression of a lower bound of the quantity occurring in Turán's Inequality.

THEOREM 13. *Let $\alpha \in \Sigma$. Then for every $\delta > |Z(\alpha')|$ the inequality*

$$\int_{b-\delta}^{b+\delta} P_n^2(x)\, d\alpha(x) \geqslant C_1(d\alpha, \delta) := \frac{\sigma(\delta)\,\delta^2}{576} > 0, \quad n = 0, 1, 2, ..., \tag{80}$$

holds, provided (77) *is valid.*

Theorem 13 makes it possible to extend certain results of Turán [17, Theorems I, II, and III] and Wiener [20, Theorem I], and to give explicit expressions for the quantities occurring in these results. The latter is just a reason to propose Problem LXXI for Turán.

THEOREM 14. *Let $\alpha \in \Sigma$ and $1 > \delta > 4\,|Z(\alpha')|$. If*

$$f_N(x) = \sum_{j=1}^{N} a_j P_{\nu_j}(x) \tag{81}$$

satisfies the gap condition

$$\nu_1 \geqslant B_1(d\alpha, \delta) := \frac{110592\pi}{\sigma(\delta/4)\,\delta^3}, \quad \nu_{j+1} - \nu_j \geqslant B_1(d\alpha, \delta), \quad j = 1, ..., N-1, \tag{82}$$

then the inequality

$$\int_{-1}^{1} f_N^2(x)\, d\alpha(x) \leqslant C_2(d\alpha, \delta) \int_{b-\delta}^{b+\delta} f_N^2(x)\, d\alpha(x) \tag{83}$$

holds for all b's satisfying (77), *where*

$$C_2(d\alpha, \delta) = \frac{147456}{\sigma(\delta/4)\,\delta^2}. \tag{84}$$

Proof. As stated in [17, p. 300], having $C_1(d\alpha, \delta)$, we can define the gap condition as follows. Assume that the numbers b and δ satisfy (77). Now put

$$f_0(x) := f_0(\delta, b; x) = \begin{cases} 1, & \text{for } x = b \\ 0, & \text{for } x \leqslant b-\delta \text{ or } x \geqslant b+\delta \\ \text{linear}, & \text{for } x \in [b-\delta, b] \\ \text{linear}, & \text{for } x \in [b, b+\delta]. \end{cases}$$

Let $E_m(f_0)$ be the deviation of best uniform approximation of f_0 in $[-1, 1]$ by polynomials of degree $\leqslant m$. Obviously $f_0 \in \mathrm{Lip}_{1/\delta}\, 1$. Thus according to Jackson's Theorem V [2, p. 147] we obtain

$$E_m(f_0) \leqslant \frac{\pi}{2m\delta}.$$

Hence, if δ is fixed and b varies, then

$$k_m(\delta) := \sup_b E_m(f_0) \leqslant \frac{\pi}{2m\delta}. \tag{85}$$

Choose $B(d\alpha, \delta)$ so large that the inequality

$$k_m(\delta) < \frac{1}{8} C_1\left(d\alpha, \frac{\delta}{4}\right) \tag{86}$$

holds whenever $m \geqslant B(d\alpha, \delta)$. By (85) it is sufficient to solve the inequality

$$\frac{\pi}{2m\delta} \leqslant \frac{1}{8} C_1\left(d\alpha, \frac{\delta}{4}\right),$$

which implies

$$m \geqslant \frac{36864\pi}{\sigma(\delta/4)\,\delta^3} := B(d\alpha, \delta).$$

Therefore, by [17, (4.7)]

$$B_1(d\alpha, \delta) = 3B(d\alpha, \delta) = \frac{110592\pi}{\sigma(\delta/4)\,\delta^3}.$$

Again, as proved in [17, p. 300], under the gap condition (82) inequality (83) holds if we take

$$C_2(d\alpha, \delta) = 16C_1(d\alpha, \delta)^{-1},$$

which implies (84). ∎

Applying Theorem 14 instead of Theorem I in [17], and using the Turán–Wiener arguments in [17] we can state the following two results, which extend Theorem III and Theorem II in [17], respectively. But we will not give detailed proofs, since the proofs are similar (see the comment in [10, p. 261]).

Corollary 15. *Let $\alpha \in \Sigma$ and $b - a > 8\,|Z(\alpha')|$. If the formal series*

$$\sum_{j=1}^{\infty} a_j P_{\nu_j}(x), \tag{87}$$

satisfying the gap condition (82), is $L^2_{d\alpha}$ Abel summable on a subinterval $[a, b]$ of $[-1, 1]$ to an $f(x)$ with $f(x) \in L^2_{d\alpha}([a, b])$, then it is $L^2_{d\alpha}$ Abel summable on $[-1, 1]$ to an $f(x)$ with $f(x) \in L^2_{d\alpha}([-1, 1])$.

COROLLARY 16. *Let $\alpha \in \Sigma$ and $b-a>8\,|Z(\alpha')|$. If a sequence $V=\{v_j\}$ has the two properties that* (a) *the sequence $\{P_{v_j}(x)\}$ spans $C[a,b]$ in the sup norm, and* (b) *omitting finitely many of the v_j's the remaining v_j's still have property* (a), *then*

$$\liminf_{j\to\infty}(v_{j+1}-v_j)<B_1\left(d\alpha,\frac{b-a}{2}\right). \tag{88}$$

4.5. *Hermite–Fejér Interpolation*

The Hermite–Fejér interpolation of $f\in C[-1,1]$ at the zeros of $P_n(d\alpha,x)$ is defined by

$$H_n(f):=H_n(f,x):=H_n(d\alpha,f):=H_n(d\alpha,f,x):=\sum_{k=1}^{n} f(x_{kn})\,A_{kn}(x),$$

where

$$A_{kn}(x):=\left[1-\frac{\omega_n''(x_{kn})}{\omega_n'(x_{kn})}(x-x_{kn})\right]l_{kn}^2(x)$$
$$:=v_{kn}(x)\,l_{kn}^2(x),\qquad k=1,2,...,n,\quad n=1,2,....$$

THEOREM 15. *Let $\alpha\in\Sigma$.*

(a) *If*

$$\lim_{n\to\infty}\|H_n(d\alpha,f)-f\|=0,\qquad \forall f\in C[-1,1], \tag{89}$$

then

$$\|P_n(d\alpha)\|=o(n^{1/2}); \tag{90}$$

(b) *if $H_n(d\alpha)$ is ρ-normal, i.e.,*

$$v_{kn}(x)\geqslant\rho>0,\qquad x\in[-1,1],\quad k=1,2,...,n,\quad n=1,2,..., \tag{91}$$

then for every $\varepsilon>0$

$$\|P_n(d\alpha)\|=o(n^{(1-\rho)/2+\varepsilon}). \tag{92}$$

Proof. (a) It is easy to check that

$$x=\sum_{k=1}^{n}x_kA_k(x)+\sum_{k=1}^{n}(x-x_k)\,l_k^2(x)=H_n(x,x)+\sum_{k=1}^{n}(x-x_k)\,l_k^2(x)$$

and

$$x^2 = \sum_{k=1}^{n} x_k^2 A_k(x) + 2 \sum_{k=1}^{n} x_k(x - x_k)\, l_k^2(x)$$
$$= H_n(x^2, x) + 2 \sum_{k=1}^{n} x_k(x - x_k)\, l_k^2(x).$$

From these identities it follows by (89) that

$$\sum_{k=1}^{n} (x - x_k)^2\, l_k^2(x) = x[x - H_n(x, x)] - \tfrac{1}{2}[x^2 - H_n(x^2, x)]$$
$$\leqslant \|x - H_n(x, x)\| + \|x^2 - H_n(x^2, x)\| = o(1),$$

which implies (90) by virtue of (37).

(b) We need a result proved by Grünwald in [7, formula (92), p. 236], which states that if H_n is ρ-normal then for every $\varepsilon > 0$

$$\sum_{k=1}^{n} |(x - x_k)\, l_k^2(x)| = O(n^{-\rho + 2\varepsilon}).$$

Clearly, this remains true provided we replace the sign "O" by "o." Thus

$$\sum_{k=1}^{n} (x - x_k)^2\, l_k^2(x) = o(n^{-\rho + 2\varepsilon}).$$

Again using (37) we obtain (92). ∎

4.6. L^2 *Version of the Principle of Contamination*

Given $p > 0$, for $\Delta \in \mathcal{M}$, define

$$\|f\|_{L^p_{d\alpha}(\Delta)} := \left\{ \int_\Delta |f(x)|^p \, d\alpha(x) \right\}^{1/p}.$$

Recently, X. Li *et al.* in [8, Theorem 2.1] proved the following result, which illustrates an L^2 version of the principle of contamination.

THEOREM F. *Suppose that* $\alpha'(x) > 0$, *a.e. on* $[-1, 1]$. *Let* $f \in L^2_{d\alpha}[-1, 1]$, f *not a polynomial, and* $\delta \in (0, 2]$. *Then*

$$\sum_{n=0}^{\infty} \left\{ \frac{\|f - S_n(d\alpha, f)\|_{L^2_{d\alpha}[a, b]}}{\|f - S_n(d\alpha, f)\|_{L^2_{d\alpha}[-1, 1]}} \right\}^2 = \infty \tag{93}$$

holds uniformly for $[a, b] \subset [-1, 1]$ *with* $b - a \geqslant \delta$.

We point out that the proof of this theorem uses only Turán's Inequality for $\alpha'(x) > 0$ a.e., given by A. Máté *et al.* [10]. Now using Turán's Inequality (45) for general measures one can extend Theorem F as follows.

THEOREM 16. *Let* $\alpha \in \Sigma$. *Let* $f \in L^2_{d\alpha}[-1, 1]$, f *not a polynomial, and* $\delta > |Z(\alpha')|$. *Then*

$$\sum_{n=0}^{\infty} \left\{ \frac{\|f - S_n(d\alpha, f)\|_{L^2_{d\alpha}(\Delta)}}{\|f - S_n(d\alpha, f)\|_{L^2_{d\alpha}[-1,1]}} \right\}^2 = \infty \tag{94}$$

holds uniformly for $\Delta \in \mathcal{M}$ *with* $|\Delta| \geq \delta$.

We leave the details of the proof to the reader.

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REFERENCES

1. R. ASKEY, Mean convergence of orthogonal series and Lagrange interpolation, *Acta Math. Acad. Sci. Hungar.* **23** (1972), 71–85.
2. E. W. CHENEY, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
3. P. ERDŐS AND P. TURÁN, On interpolation, I, *Ann. of Math. (2)* **38** (1937), 142–151.
4. P. ERDŐS AND P. TURÁN, On interpolation, III, *Ann. of Math. (2)* **41** (1940), 510–553.
5. G. FREUD, "Orthogonal Polynomials," Akad. Kiadó, Budapest, and Pergamon, Oxford, 1971.
6. G. FREUD, On Hermite–Fejér interpolation processes, *Studia Sci. Math. Hungar.* **7** (1972), 307–316.
7. G. GRÜNWALD, On the theory of interpolation, *Acta Math.* **75** (1942), 219–245.
8. X. LI, E. B. SAFF, AND Z. SHA, Behavior of best L^p polynomial approximants on the unit interval and on the unit circle, *J. Approx. Theory* **63** (1990), 170–190.
9. A. MÁTÉ, P. NEVAI, AND V. TOTIK, Necessary conditions for weighted mean convergence of Fourier series in orthogonal polynomials, *J. Approx. Theory* **46** (1986), 314–322.
10. A. MÁTÉ, P. NEVAI, AND V. TOTIK, Strong and weak convergence of orthogonal polynomials, *Amer. J. Math.* **109** (1987), 239–282.
11. P. NEVAI, "Orthogonal Polynomials," Memoirs of the Amer. Math. Soc., Vol. 213, Amer. Math. Soc., Providence, RI, 1979.
12. P. NEVAI, Solution of Turán's problem on divergence of Lagrange interpolation in L^p with $p > 2$, *J. Approx. Theory* **43** (1985), 190–193.
13. P. NEVAI, Géza Freud, Orthogonal polynomials and Christoffel functions: A case study, *J. Approx. Theory* **48** (1986), 3–167.
14. P. NEVAI AND P. VÉRTESI, Mean convergence of Hermite–Fejér interpolation, *J. Math. Anal. Appl.* **105** (1985), 26–58.

15. Y. G. SHI, On critical order of Hermite–Fejér type interpolation, *in* "Progress in Approximation Theory" (P. Nevai and A. Pinkus, Eds.), pp. 761–766, Academic Press, New York, 1991.
16. G. SZEGŐ, "Orthogonal Polynomials," Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, RI, 1939.
17. P. TURÁN, On orthogonal polynomials, *Anal. Math.* **1** (1975), 297–311.
18. P. TURÁN, On some open problems of approximation theory, *J. Approx. Theory* **29** (1980), 23–85.
19. P. VÉRTESI, On sums of Lebesgue function type, *Acta Math. Acad. Sci. Hungar.* **40** (1982), 217–227.
20. N. WIENER, A class of gap theorems, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **3** (1934), 367–372.